3

ON THE DERIVATION OF THE COUPLED WAVE EQUATIONS IN THE MAGNETO-IONIC THEORY

S. K. BANERJEE

UNIVERSITY COLLEGE OF SCIENCE, CALCUTTA.

(*Received November* 4, 1964)

ABSTRACT. It has been shown that there exists ambiguity of sign in the coupled wave equations as deduced by Saha, Banerjea and Guha (1951). It has also been shown that the same ambiguity of sign appears when the coupled wave equations as given by Budden (1952) are expressed in the form given by Saha *et al.*

INTRODUCTION

It has been shown in our previous paper (1964) that the coupled wave equations are given by :

$$\ddot{V}_1+(q^2_x-\dot{\phi}_1^2)V_1 = 2\dot{\phi}_1\dot{W}_1+\ddot{\phi}_1 W_1 \text{ (for the } X\text{-mode)} \quad \ldots \quad (1)$$

$$\ddot{W}_1+(q_0^2-\dot{\phi}^2_1)W_1 = -2\dot{\phi}_1\dot{V}_1-\ddot{\phi}_1 V_1 \text{ (for the 0-mode)} \quad \ldots \quad (2)$$

$$\ddot{W}_2+(q^2_x-\dot{\phi}_2^2)W_2 = -2\dot{\phi}_2\dot{V}_2-\ddot{\phi}_2 V_2 \text{ (for the } X\text{-mode)} \quad \ldots \quad (1.1)$$

$$\ddot{V}_2+(q_0^2-\dot{\phi}_2^2)V_2 = 2\dot{\phi}_2\dot{W}_2+\ddot{\phi}_2 W_2 \text{ (for the 0-mode)} \quad \ldots \quad (2.1)$$

where,

$$V_1 = E_x\cos\phi_1+jE_y\ \sin\phi_1 = \frac{E_x+j\rho_1E_y}{\sqrt{1+\rho_1^2}} \text{ (for the } X\text{-mode)} \quad \ldots \quad (3)$$

$$W_1 = -E_x\sin\phi_1+jE_y\cos\phi_1 = \frac{-\rho_1E_x+jE_y}{\sqrt{1+\rho_1^2}} \text{ (for the 0-mode)} \quad \ldots \quad (4)$$

$$V_2 = E_x\cos\phi_2+jE_y\ \sin\phi_2 = \frac{E_x+j\rho_2E_y}{\sqrt{1+\rho_2^2}} \text{ (for the 0-mode)} \quad \ldots \quad (4.1)$$

$$W_2 = -E_x\sin\phi_2+jE_y\cos\phi_2 = \frac{-\rho_2E_x+jE_y}{\sqrt{1+\rho^2_2}} \text{ (for the } X\text{-mode)} \quad \ldots \quad (3.1)$$

$$\left.\begin{aligned}
q^2_x &= 1-\frac{r}{\beta'+\rho_1\omega_z} = K_1\cos^2\phi_1+K_2\sin^2\phi_1-2L\sin\phi_1\cos\phi_1\\
q_0{}^2 &= 1-\frac{r}{\beta'+\rho_2\omega_z} = K_1\cos^2\phi_2+K_2\sin^2\phi_2-2L\sin\phi_2\cos\phi_2\\
q_0{}^2 &= 1-\frac{r}{\beta'+\rho_2\omega_z} = K_1\sin^2\phi_1+K_2\cos^2\phi_1+2L\sin\phi_1\cos\phi_1\\
q^2_x &= 1-\frac{r}{\beta'+\rho_1\omega_z} = K_1\sin^2\phi_2+K_2\cos^2\phi_2+2L\sin\phi_2\cos\phi_2
\end{aligned}\right]\quad\ldots\quad(5)$$

$$\left.\begin{aligned}
\rho_1 &= \tan\phi_1 = G-\sqrt{1+G^2}\\
\rho_2 &= \tan\phi_2 = G+\sqrt{1+G^2}
\end{aligned}\right]\quad\ldots\quad(6)$$

$$\dot{\phi}_1 = \frac{\partial\rho_1/\partial u}{1+\rho^2{}_1},\quad \dot{\phi}_2 = \frac{\partial\rho_2/\partial u}{1+\rho^2{}_2},\quad u = \frac{2\pi}{\lambda}z\quad\ldots\quad(7)$$

$$G = \frac{K_1-K_2}{2L} = \frac{\omega_x{}^2}{2\omega_z(r-\beta')}$$

$$K_1 = 1-\frac{\beta'^2-r\beta'-\omega_x{}^2}{C'}r,\quad K_2 = 1-\frac{r(\beta'^2-r\beta')}{C'}$$

$$L = \frac{r(r-\beta')\omega_z}{C'},\quad C' = \beta'(\beta'^2-\omega^2)-r(\beta'^2-\omega_z{}^2)$$

$$\omega_x = \omega\sin\theta,\quad \omega_z = \omega\cos\theta,\quad \omega = p_H/p,\ p_H = eH/mc$$

$$\beta' = 1-j\frac{\nu}{p},\quad r = \frac{p_0{}^2}{p^2} = \frac{4\pi Ne^2}{mp^2}$$

$$\dot{V} = \frac{dV}{du},\quad \dot{W} = \frac{dW}{du}.$$

AMBIGUITY OF SIGN IN THE COUPLED WAVE EQUATIONS OF SAHA *et al.*

Using (6), (4), (4.1), and (6), (3), (3.1) we have:

$$\left.\begin{aligned}
V_1 &= \pm W_2\\
W_1 &= \pm V_2
\end{aligned}\right]\quad\ldots\quad(8)$$

From eqn. (8) we can make the following combinations :

$$\begin{pmatrix} V_1 = W_2 \\ W_1 = V_2 \end{pmatrix} \qquad \dots \quad (8.1)$$

$$\begin{pmatrix} V_1 = -W_2 \\ W_1 = -V_2 \end{pmatrix} \qquad \dots \quad (8.2)$$

$$\begin{pmatrix} V_1 = W_2 \\ W_1 = -V_2 \end{pmatrix} \qquad \dots \quad (8.3)$$

$$\begin{pmatrix} V_1 = -W_2 \\ W_1 = V_2 \end{pmatrix} \qquad \dots \quad (8.4)$$

It can be easily shown from (6) that

$$\phi_2 - \phi_1 = n\,\frac{\pi}{2} \qquad \dots \quad (9)$$

where $\quad n = 0, \ \pm 1, \ \pm 2, \ \pm 3 \dots$

For $\quad n = 0$

$$\phi_2 = \phi_1 \qquad \dots \quad (9.1)$$

Using (9.1) and (5) it can be shown that the electron number density N must be zero. Similarly for $n = \pm 2, \pm 4, \pm 6 \dots$ using (5) and (9) we get $N = 0$. We can therefore exclude the cases $n = 0, \pm 2, \pm 4 \dots$. However for $n = \pm 1, \pm 3, \dots$ eqns (5) are satisfied by (9). It can be shown using (3), (3.1), (4), (4.1) and (9) that we can take the combination (8.1) only when the following two equations hold good *simultaneously* :

$$\phi_2 - \phi_1 = n\,\frac{\pi}{2} \quad \text{where } n = 1, 5, 9, \dots, -3, -7, -11, \dots$$

$$\phi_2 - \phi_1 = n\,\frac{\pi}{2} \quad \text{where } n = -1, -5, -9, \dots, 3, 7, 11, \dots$$

which is absurd. Hence we reject combination (8.1). Similarly combination (8.2) must be rejected. For $n = -1, -5, -9 \dots +3, +7, +11 \dots$ using (3), (3.1), (4), (4.1) and (9) we get the combination (8.3). Similarly for $n = -3, -7, -11 \dots, +1, +5 +9 \dots$ we get the combination (8.4). Hence using (8.3) eqns. (1), (1.1), (2) and (2.1) can be combined into :

$$\ddot{V}_1 + (q_X^2 - \dot{\phi}^2)V_1 = -2\dot{\phi}\,\dot{V}_2 - \ddot{\phi}V_2 \ \text{(for the } X\text{-mode)} \qquad \dots \quad (1.2)$$

$$\ddot{V}_2 + (q_0^2 - \dot{\phi}^2)V_2 = 2\dot{\phi}\dot{V}_1 + \ddot{\phi}V_1 \ \text{(for the 0-mode)} \qquad \dots \quad (2.2)$$

and using (8.4), eqns. (1), (1.1), (2) and (2.1) can be combined into :

$$\ddot{V}_1+(q_X{}^2-\dot{\phi}^2)V_1 = 2\dot{\phi}\ \dot{V}_2+\ddot{\phi}V_2 \text{ (for the } X\text{—mode)} \qquad \dots (1.3)$$

$$\ddot{V}_2+(q_0{}^2-\dot{\phi}^2)V_2 = -2\dot{\phi}\ \dot{V}_1-\ddot{\phi}V_1 \text{ (for the 0-mode)} \qquad \dots (2.3)$$

Comparing (1.2) and (2.2) with (1.3) and (2.3), we find that there is an ambiguity of signs in these coupled wave-equations.

STATEMENT OF THE COUPLED WAVE-EQUATIONS AS GIVEN BY BUDDEN

In Budden's treatment (1952) E_1, E_2 and E_3 were the components of the electric vector E of the wave, where E_1 was horizontal and perpendicular to the earth's magnetic field, E_2 was parallel to the horizontal component of the earth's magnetic field and E_3 was directed vertically upwards and the following notations were used :

$$\left.\begin{aligned} \pi_0 &= E_{10}\sqrt{1-R_0{}^2} \\ \pi_X &= E_{1X}\sqrt{1-R_X{}^2} \\ \psi &= \frac{j}{R_0{}^2-1}\ \frac{dR_0}{dh} \end{aligned}\right] \qquad \dots (10)$$

$$\left.\begin{aligned} \left.\begin{matrix} R_0 \\ R_X \end{matrix}\right] &= -\frac{jY_T{}^2}{2Y_L(1-X-jZ)} \pm j\sqrt{1+\frac{Y_T{}^4}{4Y_L{}^2(1-X-jZ)^2}} \\ M^2 &= 1-\frac{X}{1-jZ-jY_L/R} \end{aligned}\right] \qquad \dots (11)$$

$$R = \frac{E_2}{E_1} \qquad \dots (12)$$

$$\left.\begin{aligned} X &= \frac{4\pi Ne^2}{\epsilon_0 mp^2} \\ Y &= p_H/p \\ Z &= \nu/p \\ p_H &= \frac{\mu_0 eH}{m} \end{aligned}\right] \qquad \dots (13)$$

Y_L = Vertical component of Y.
Y_T = horizontal component of Y.
N = electron number density.
ν = electronic collisional frequence.
e, m = charge and mass of an electron
H = strength of the earth's magnetic field.

ε_0, μ_0 = electric and magnetic permittivities of free space. With these notations, it was shown by Budden that the coupled wave-equations would be given by :

$$\frac{d^2\pi_0}{dh^2} + (K^2M_0^2-\psi^2)\pi_0 = -\pi_X \frac{d\psi}{dh} - 2\psi \frac{d\pi_X}{dh} \quad \ldots \quad (14)$$

$$\frac{d^2\pi_X}{dh^2}+(K^2M^2{}_X-\psi^2)\,\pi_X = \pi_0 \frac{d\psi}{dh} + 2\psi \frac{d\pi_0}{dh} \quad (15)$$

INCORRECT ASSOCIATION OF THE EXPRESSIONS FOR R AND M^2 WITH THE O- AND X-MODES

We shall consider the following equations of motion of an electron of charge e as given by Budden :

$$\frac{\epsilon_0 X}{4\pi} E_1 = \left[\frac{Y_T^2}{1-X-jZ} -(1-jZ)\right]P_1 - jY_LP_2$$

$$\frac{\epsilon_0 X}{4\pi} E_2 = -(1-jZ)P_2+jY_LP_1 \quad (16)$$

As the axes of the co-ordinate system in Budden's treatment are not clearly defined we shall compare eqn. (16) with the following equations which are derived with reference to the co-ordinate system as shown in Fig. 1.

$$\frac{\epsilon_0 X}{4\pi} E_y = -(1-jZ)P_y+jY_LP_x$$

$$\frac{\epsilon_0 X}{4\pi} E_x = \left[\frac{Y_T^2}{(1-X-jZ)}-(1-jZ)\right]P_x-jY_LP_y \quad (17)$$

The comparison shows that E_1 is along ox, and E_2 is along Oy of Fig. 1. Hence eqn. (12) can be written as :

$$R = \frac{E_y}{E_x} \quad (12.1)$$

Using eqns. (12.1) and (11) we get for the 0-mode :

$$\left(\frac{E_x}{E_y}\right)_0 = -\frac{j}{Y_L}\left[\frac{\frac{1}{2}Y^2{}_T}{(1-X-jZ)} + \sqrt{Y^2{}_L + \frac{1/4\ Y^4{}_T}{(1-X-jZ)^2}}\right]$$

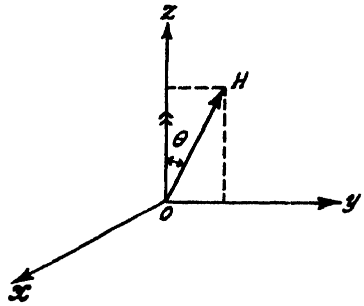

Fig. 1. Coordinate System (present paper)
OZ→ direction of wave propagation
OH→ direction of the Earth's magnetic field in the plane yz

This equation shows that the expression for the wave-polarization which Budden associated with the 0-mode is actually that for the X-mode in the ray treatment (Ratcliffe, 1962). Again using (11) we get for the 0-mode :

$$M_0^2 = 1 - \frac{X}{1 - jZ - \frac{Y_T^2}{2(1-X-jZ)} - \sqrt{Y^2_L + \frac{Y^4_T}{4(1-X-jZ)^2}}}$$

This equation shows that the expression for the square of the complex refractive index which Budden associated with the 0-mode agrees with that for the X-mode in the ray treatment (Ratcliffe, 1962). Thus we find that what Budden (1952) called the 0-mode in his wave-treatment is actually the x-mode in the same treatment and is identified with that for the X-mode in the ray-treatment. Hence eqn. (11) should be written as follows :

$$\left.\begin{aligned} \frac{R_x}{R_0}\bigg] &= -\frac{jY^2_T}{2Y_L(1-X-jZ)} \pm j\sqrt{1 + \frac{Y_T^4}{4Y^2_L(1-X-jZ)^2}} \\ M^2 &= 1 - \frac{X}{1 - jZ - jY_L/R} \end{aligned}\right] \quad \dots (11.1)$$

The question now naturally arises about the correct labelling of the coupled wave-equations (14) and (15). We have given below the reasons that eqns. (14) and (15) were *correctly* labelled.

Budden (1952) deduced the coupled wave-equations from the following :

$$\frac{d^2E_1}{dh^2} + \frac{K^2}{\epsilon_0} D_1 = 0, \quad \frac{d^2E_2}{dh^2} + \frac{K^2}{\epsilon_0} D_2 = 0$$

where

$$E_1 = E_{10}+E_{1X},\ E_2 = E_{20}+E_{2X}$$

$$R_0 = \frac{E_{20}}{E_{10}}\ ,\ R_X = \frac{E_{2X}}{E_{1X}}$$

$$D_1 = \epsilon_0[M_0{}^2E_{10}+M_X{}^2E_{1X}]$$

$$D_2 = \epsilon_0[M_0{}^2E_{20}+M_X{}^2E_{2X}$$

It is to be noted that the expressions on the right hand side of Eqn. (11) did *not* enter in the method of derivation of the coupled wave-equations given in (14) and (15) by Budden (1952). Hence these equations (14) and (15) were correctly labelled. However whenever the values of R_0, R_X, $M_o{}^2$ and $M_X{}^2$ are to be substituted in eqns. (14) and (15) or in solutions of these equations, the equations (11.1) *must* be used.

BUDDEN'S COUPLED WAVE-EQUATIONS EXPRESSED IN THE FORM GIVEN BY SAHA et al

We start from the equation given in our previous paper (1964) :

$$D_y = q^2E_y = K_2E_y+jLE_x \qquad \ldots \quad (18)$$

Using (18) and (5), it can be easily shown that for the 0-mode :

$$\left(\frac{E_x}{E_y}\right)_0 = -j\rho_1 \qquad \ldots \quad (19)$$

Similarly it can be shown for the X-mode.

$$\left(\frac{E_x}{E_y}\right)_\times = -j\rho_2 \qquad \ldots \quad (19.1)$$

Equations (19) and (19.1) are referred to the co-ordinate system of Fig. 2.

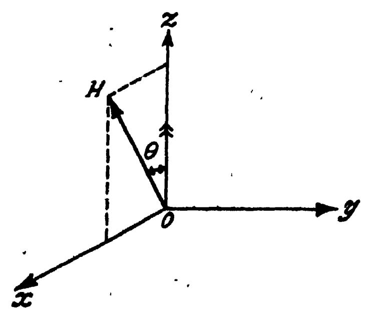

Fig. 2. Co-ordinate System (Saha *et al*)
OZ → direction of wave propagation
OH → direction of the Earth's magnetic field in the plane xz

Comparing the co-ordinate systems of Figs. 1 and 2 and using the equations (3), (4.1), (6), (7), (10), (11.1), (12), (12.1), (19) and (19.1) we have

$$\left.\begin{aligned} \pi_X = \pm jV_1, \quad \pi_0 = \pm jV_2 \\ \psi/K = \dot{\phi}_1 = \dot{\phi}_2 = \dot{\phi} \end{aligned}\right] \quad \ldots \quad (20)$$

From eqn. (20) we can make the following combinations :

$$\begin{pmatrix} \pi_X = jV_1 \\ \pi_0 = jV_2 \end{pmatrix} \quad \ldots \quad (20.1)$$

$$\begin{pmatrix} \pi_X = -jV_1 \\ \pi_0 = -jV_2 \end{pmatrix} \quad \ldots \quad (20.2)$$

$$\begin{pmatrix} \pi_X = jV_1 \\ \pi_0 = -jV_2 \end{pmatrix} \quad \ldots \quad (20.3)$$

$$\begin{pmatrix} \pi_X = -jV_1 \\ \pi_0 = jV_2 \end{pmatrix} \quad \ldots \quad (20.4)$$

It can be easily shown using either (20.1) or (20.2) and $\psi/K = \dot{\phi}$, $q^2 = M^2$ that the equations (14) and (15) can be transformed into (2.3) and (1.3) respectively. Similarly using either (20.3) or (20.4), the equations (14) and (15) can be transformed into (2.2) and (1.2) respectively.

We have shown elsewhere (1964) that the wave-equations of Saha *et al.* were incorrectly labelled i.e., the wave-equation which they associated with the 0-mode should actually be that for the X-mode and *vice-versa.* Moreover we have shown in the present paper that the wave-equations were correctly labelled by Budden (1952). But according to Kelso (1953) the coupled wave-equations, as deduced by Saha *et al* (1951) are identical with those given by Budden (1952). The discrepancy is due to the fact that Kelso (1953) incorrectly associated the wave function π_0 used by Budden for the 0-mode with the wave function V which Saha *et al.* used for the 0-mode where V (which is the same as V_1 in the present paper) is *actually* the wave function for the X-mode.

ACKNOWLEDGEMENT

The author is deeply indebted to Prof. S. R. Khastgir, D.Sc., F.N.I., for his keen interest and constant guidance.

REFERENCE

Banerjee, S. K. and Khastgir, S. R., 1964, *Ind. Jour. Phys.* **38**, 347.

Budden, K. G., 1952, *Proc. Roy. Soc. Series A*, **215**, 215.

Kelso, J. M. 1953, *J. Geophys. Res.*, **58**, 431.

Ratcliffe, J. A. 1962, The Magneto-Ionic Theory and its Applications to the Ionosphere (Cambridge University Press), p. 19.

Saha, M. N., Banerjea, B. K. and Guha, U. C., 1951, *Proc. Nat. Inst. Sci.* (India) **17**, 205.